Eur. Phys. J. A **18**, 211–214 (2003)
DOI 10.1140/epja/i2002-10300-9

THE EUROPEAN PHYSICAL JOURNAL A

Phase diagram of the three-dimensional NJL model

C.G. Strouthos[a]

Department of Physics, University of Wales Swansea, Singleton Park, Swansea SA2 8PP, UK

Received: 30 September 2002 /
Published online: 22 October 2003 –

Abstract. With the exception of confinement the three-dimensional Nambu–Jona-Lasinio (NJL_3) model incorporates many of the essential properties of QCD. We discuss the critical properties of the model at non-zero temperature T and/or non-zero chemical potential μ. We show that the universality class of the thermal transition is that of the $d = 2$ classical spin model with the same symmetry. We provide evidence for the existence of a tricritical point in the (μ, T)-plane. We also discuss numerical results by Hands *et al.* which showed that the system is critical for $\mu > \mu_c$ and the diquark condensate is zero.

PACS. 71.10.Fd Lattice fermion models (Hubbard model, etc.) – 05.70.Fh Phase transitions: general studies

1 Introduction

Phase transitions in QCD at non-zero temperature and/or non-zero baryon density have been studied intensively over the last decade both analytically and numerically. However, since the problem of chiral symmetry breaking and its restoration is intrinsically non-perturbative, the number of available techniques is limited and most of our knowledge about the phenomenon comes from lattice simulations. Because of the complexity of QCD with dynamical fermions, studies so far have been done on lattices with modest size and in various cases the results are distorted by finite size and discretization effects.

The NJL model has been proved to be an interesting and tractable laboratory to study chiral phase transitions both numerically by means of lattice simulations and analytically in the form of large-N_f expansions [1–9]. The Lagrangian density of the $U(1)$-symmetric model is

$$\mathcal{L} = \bar{\psi}_i(\partial\!\!\!/ + m + \sigma + i\gamma_5\pi)\psi_i + \frac{N_f}{2g^2}(\sigma^2 + \pi^2), \quad (1)$$

where the index i runs over N_f fermion flavors. There are several features which make this model interesting for the modelling of strong interactions: i) The spectrum of excitations contains both "baryons" and "mesons", namely the elementary fermions f and the composite $f\bar{f}$ states [10]. ii) For sufficiently strong coupling $g^2 > g_c^2$ it exhibits spontaneous chiral symmetry breaking implying dynamical generation of a fermion mass m_f, the pion field π being the associated Goldstone boson. iii) For $2 < d < 4$ there is an interacting continuum limit at a critical value of the coupling, which for $d = 3$ has a numerical value $g_c^2/a \approx 1.0$ in the large-N_f limit if a lattice regularisation is employed [2]. There is a renormalisation group UV fixed point at $g^2 = g_c^2$, signalled by the renormalisability of the $1/N_f$ expansion [1], entirely analogous to the Wilson-Fisher fixed point in scalar field theory. iv) Numerical simulations with baryon chemical potential $\mu \neq 0$ show qualitatively correct behavior, in that the onset of matter occurs for μ of the same order as the constituent-quark scale m_f [3], rather than for $\mu \approx m_\pi/2$, which happens in gauge theory simulations with a real measure $\det(M^\dagger M)$ because of the presence of a baryonic pion in the spectrum. This makes NJL_3 an ideal arena in which to test strongly interacting thermodynamics.

In sect. 2 we discuss the universality of the $T \neq 0$ transition [4,5]. In sect. 3 we present results from a study of the phase diagram in the (μ, T)-plane [8] which support the existence of a tricritical point on the line that separates the chirally broken from the chirally symmetric phases. We also discuss numerical results which support the non-existence of a diquark condensate for $\mu > \mu_c$ [9].

2 Universality at non-zero temperature

Although there is little disagreement that the chiral phase transition in QCD with two massless fermions is second order, no quantitative work or simulations have been done that decisively determine its universality class. Universality arguments are very appealing due to their beauty and simplicity. In essence they can be phrased as follows: At finite T phase transitions, the correlation length diverges in the transition region and the long-range behavior is that

[a] e-mail: C.G.Strouthos@swansea.ac.uk

Table 1. Summary of FSS results and comparison with $2d$ Ising and mean-field scaling behavior.

Exponents	FSS	$Z(2)$	MF
ν	1.00(3)	1	0.5
β_m/ν	0.12(6)	0.125	1
γ/ν	1.66(9)	1.75	2

of the $(d-1)$ classical spin model with the same symmetry, because the IR region of the system is dominated by the zero mode of the bosonic field and the contribution of non-zero modes does not affect the critical singularities but can be absorbed into non-critical, non-universal aspects of the transition. Consequently, fermions, which satisfy anti-periodic boundary conditions, and do not have zero modes, are expected to decouple from the scalar sector. In another language, the classical thermal fluctuations whose energy is $O(K_B T)$ dominate over quantum fluctuations with energy $O(\hbar\omega)$ for soft modes of the field and the effective $(d-1)$-dimensional theory for the bosonic fields near T_c is a classical statistical theory. A possible loophole to this standard scenario is that the mesons are composite $f\bar{f}$ states and their size and density increase as $T \to T_c$. Therefore, if the transition region can be described as a system of highly overlapping composites, the violation of the bosonic character of the mesons may be maximal and the fermions become essential degrees of freedom irrespective of how heavy they are. At leading order in $1/N_f$ the model has a second-order phase transition at $T_c = \frac{m_f}{2\ln 2}$ [4] with Landau-Ginzburg mean-field scaling. We studied the critical behavior of the Z_2-symmetric model with finite N_f by performing lattice simulations in the vicinity of the critical point. The temporal lattice size was $L_t = 6$ and the spatial size varied from $L_s = 18$ to 50. The expectation value of the auxiliary sigma field ($\Sigma \equiv \langle\sigma\rangle$) serves as a convenient order parameter for the theory's critical point. We simulated the model exactly for $N_f = 12$ with the Hybrid Monte Carlo method. The staggered fermion lattice action and further details concerning the algorithm can be found in [2]. By using the finite-size scaling (FSS) method we extracted the critical exponents. The results which are summarized in table 1 support the dimensional reduction scenario, because the values of the exponents are in good agreement with those of the $2d$ Ising model rather than the mean-field theory ones.

Next we tried to understand how the large-N_f mean-field theory prediction reconciles with the dimensional reduction and universality arguments. The answer is that the large-N_f description has its applicability region. As we discuss in detail for a Yukawa theory in [5] the phenomenon which leads to an apparent contradiction is the suppression of the width of the non–mean-field critical region by a power of $1/N_f$. The $2d$ Ising critical behavior sets in when $T \gg m_\sigma(T)$ ($m_\sigma(T)$ is the thermal mass of the σ meson). If the cutoff $\Lambda \gg T$, the renormalized self-interaction coupling $\lambda(T)$ in the large-N_f limit is close to the IR fixed point of the Yukawa theory and is given by $\lambda(T) \sim T^{4-d}/N_f$ for $2 < d < 4$. The mean-field approximation breaks down because of self-inconsistency when the value of the coupling of the $(d-1)$-dimensional scalar theory $\lambda_{d-1} = T\lambda(T)$ on the scale $m_\sigma^{5-d}(T)$ (the power $d-5$ comes from dimensional analysis) is not small anymore. Therefore, for $d = 3$ the Ginzburg criterion for the applicability of the mean-field scaling is given by $m_\sigma(T) \gg T/\sqrt{N_f}$. This scenario was verified numerically in [5].

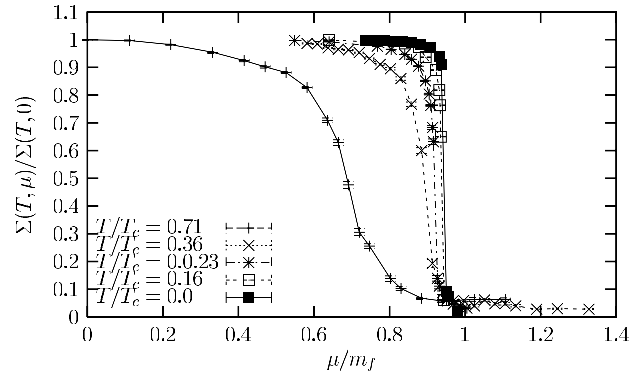

Fig. 1. $\Sigma(T,\mu)/\Sigma(T,0)$ *vs.* μ/m_f at different values of T.

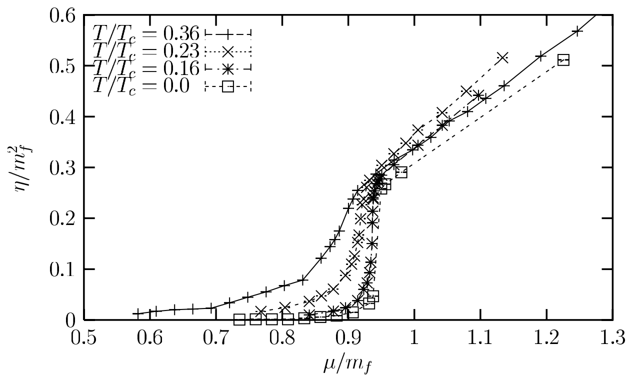

Fig. 2. $\eta(\mu)/m_f$ *vs.* μ/m_f at different values of T.

Additional evidence in favor of the dimensional reduction scenario was produced in studies of the $U(1)$-symmetric NJL$_3$ model [6]. Both analytical and numerical results showed that its phase structure at $T \neq 0$ is the same as the $2d$ XY model. It has two different chirally symmetric phases, one critical and one with finite correlation length, separated by a Berezinskii-Kosterlitz-Thouless transition.

3 Results at non-zero chemical potential

The action of the NJL model remains real even after the introduction of non-zero chemical potential μ, which means we can study the physics of the high-density regime using standard Monte Carlo techniques. In the presence of a Fermi surface with Fermi momentum p_F, the creation of $f\bar{f}$ pairs with zero net momentum is suppressed, because one can only create particles with $p > p_F$. So as the fermion number density $\eta(\mu)$ grows, the chiral symmetry breaking is suppressed. The large-N_f description of the $\mu \neq 0$ chiral phase transition predicts a first-order

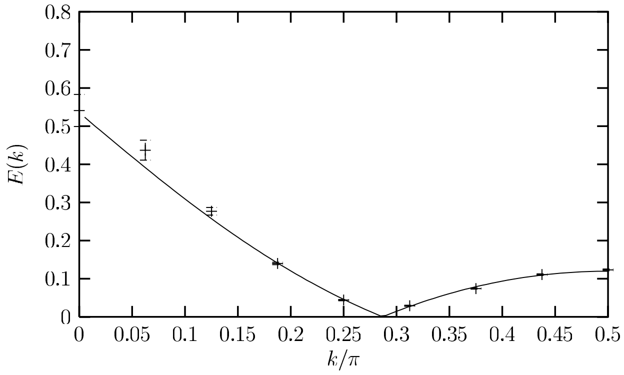

Fig. 3. Fermion dispersion relation at $\mu = 0.8$.

transition for $T = 0$ and a continuous transition for $T > 0$ [4]. Furthermore, the critical value of the chemical potential μ_c is equal to the value of the fermion mass at $\mu = 0$, which indicates that materialization of the fermion itself drives the symmetry restoration transition. Interactions as expected decrease μ_c below the mean-field result [3]. Work by Stephanov [7] suggests that any non-zero density simulation which incorporates a real path integral measure proportional to $\det(MM^\dagger)$ is doomed to failure due to the formation of a light baryonic pion from a quark q and a conjugate quark q^c. The NJL model, however, does not exhibit such a pathology, because the realization of the Goldstone mechanism in this model is fundamentally different from that in QCD. In the NJL model the Goldstone mechanism is realized by a pseudoscalar channel pole formed from disconnected diagrams and the connected diagrams yield a bound state of mass $\approx 2m_f$. This implies the absence of a light qq^c state.

As expected, our simulations of the Z_2-symmetric NJL_3 with $N_f = 4$ [8] did not provide any evidence for the existence of a nuclear liquid-gas transition at $\mu < \mu_c$. It was shown in [11] that in the NJL model there is no saturation density for stable matter. In order to get the saturation features the authors of [11] introduced a chirally invariant scalar-vector interaction term which cures the binding problem. Furthermore, our results showed that the second-order nature of the $T \neq 0$, $\mu = 0$ transition remains stable down to low T and large μ. In fig. 1 we plot the normalized order parameter $\Sigma(T,\mu)/\Sigma(T,0)$ as a function of μ/m_f at different values of T. It is clear from the shapes of these curves that the transition becomes sharper as we decrease the temperature. In fig. 2 we plot the normalized fermion number density as a function of the chemical potential at different values of T. In the limit $T \to 0$ we see that the fermion density is strongly suppressed before the transition and then jumps discontinuously. By performing detailed finite-size scaling analysis which allowed us to distinguish between second-order and weak first-order transitions we showed that the tricritical point lies on the section of the phase boundary defined by $T/T_c \leq 0.23$, $\mu/\mu_c \geq 0.97$ [8]. This result shows that higher-order corrections in the $1/N_f$ expansion for the nature and the location of the transition points in the phase diagram are small in this model.

It is well known that at high density the diquark condensate is non-zero in models of strongly interacting matter which either assume that the interaction between quarks is due to one-gluon exchange [12], or by using effective four-fermion vertices resulting from the presence of instantons in the QCD vacuum [13]. Unfortunately, theoretical studies of color superconductivity are limited to perturbative and self-consistent methods, because of the notorious "sign problem" in QCD. Hands *et al.* studied numerically the $(SU(2) \otimes SU(2))$-symmetric NJL_3 and found no evidence for a condensate $\langle qq \rangle \neq 0$ in the model's high-density phase. Their results with a non-zero diquark source are consistent with a critical behavior $\langle qq(j) \rangle \propto j^{\frac{1}{\delta}}$ throughout the dense phase with δ falling in the range 3–5 for the μ values studied. This suggests that the model is a two-dimensional superfluid as first described by Kosterlitz and Thouless for thin films of ^{4}He but with the universality class determined by the presence of massless relativistic fermions. Results for the dispersion relation $E(k)$ in the spin-$\frac{1}{2}$ sector are shown in fig. 3. For $k < k_F$ the lowest excitations vacate states in the Fermi sea, and hence are "hole-like", whereas for $k > k_F$, excitations add quarks to the system and are "particle-like". There is no sign for any discontinuity on the Fermi surface characteristic of a BCS gap $\Delta \neq 0$ in NJL_3. However, recent numerical results provided evidence that $\langle qq \rangle \neq 0$ in NJL_4 [14].

4 Summary

We discussed the basic features of the phase diagram of NJL_3. The universality class of the $T \neq 0$ transition is that of the $d = 2$ classical spin model with the same symmetry. The non-trivial critical region of the Z_2-symmetric model is suppressed by a factor $1/\sqrt{N_f}$. The non-zero density phase transition is strongly first order and the lattice simulations provided evidence for the existence of a tricritical point on the critical line at small T and large μ. The simulations showed no evidence for the existence of non-zero diquark condensate at $T = 0$ and $\mu > \mu_c$. The results are consistent with a critical behavior throughout the dense phase, suggesting that the model is a two-dimensional relativistic superfluid.

Support by the Leverhulme Trust is kindly acknowledged.

References

1. B. Rosenstein, B.J. Warr, S.H. Park, Phys. Rep. **205**, 59 (1991).
2. S.J. Hands, A. Kocić, J.B. Kogut, Ann. Phys. (N.Y.) **224**, 29 (1993).
3. S.J. Hands, S. Kim, J.B. Kogut, Nucl. Phys. B **442**, 364 (1995); I.M. Barbour, S.J. Hands, J.B. Kogut, M.-P. Lombardo, S.E. Morrison, Nucl. Phys. B **557**, 327 (1999).
4. K.G. Klimenko, Z. Phys. C **37**, 457 (1988); B. Rosenstein, B.J. Warr, S.H. Park, Phys. Rev. D **39**, 3088 (1989); S.J. Hands, A. Kocić, J.B. Kogut, Nucl. Phys. B **390**, 355 (1993); A. Kocić, J.B. Kogut, Nucl. Phys. B **455**, 229 (1995).

5. J.B. Kogut, M.A. Stephanov, C.G. Strouthos, Phys. Rev. D **58**, 096001 (1998).
6. E. Babaev, Phys. Lett. B **497**, 323 (2001); S.J. Hands, J.B. Kogut, C.G. Strouthos, Phys. Lett. B **515**, 407 (2001).
7. Misha A. Stephanov Phys. Rev. Lett. **76**, 4472 (1996).
8. J.B. Kogut, C.G. Strouthos, Phys. Rev. D **63**, 054502 (2001).
9. S.J. Hands, B. Lucini, S.E. Morrison, Phys. Rev. D **65**, 036004 (2002).
10. S.J. Hands, J.B. Kogut, C.G. Strouthos, Phys. Rev. D **65**, 114507 (2002); Nucl. Phys. B **645**, 321 (2002), hep-lat/0208030.
11. V. Koch, T.S. Biro, J. Kunz, U. Mosel, Phys. Lett. B **185**, 1 (1987). M. Buballa, Nucl. Phys. A **609**, 519 (1996).
12. B.C. Barrois, Nucl. Phys. B **129**, 390 (1977); D. Bailin, A. Love, Phys. Rep. **107**, 390 (1984).
13. R. Rapp, T. Schäfer, E.V. Shuryak, M. Velkovsky, Phys. Rev. Lett. **81**, 53 (1998); M. Alford, K. Rajagopal, F. Wilczek, Phys. Lett B **422**, 247 (1998).
14. S.J. Hands, D.N. Walters, Phys. Lett. B **548**, 196 (2002), hep-lat/0209140.